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Inequalities and variational principles in double-diffusive turbulence

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An inequality pertaining to the energetics of the boundary layer in turbulent pipe flow and turbulent thermal convection is generalized for the double-diffusive convection problem, where a semi-infinite layer of cold, fresh and light water overlies another hot, salty and dense layer. The smallest possible salt/heat-flux ratio equals the ratio of the square roots of the respective diffusivities. The bound is asymptotically realizable according to a variational principle. A bound on the relative fluxes is predicted when another solute is added ('multiple diffusion').

1. Introduction

The statistically steady solutions of the Navier–Stokes equations for turbulent pipe flow or turbulent thermal convection are highly non-unique (degeneracy), and a thermodynamically based variational principle has therefore been proposed to select the realizable solution (Stern 1980, hereinafter cited as II). The dynamically possible class of solutions provide the constraints, one of which is an 'inequality pertaining to the energetics of the boundary layer' (Stern 1979, hereinafter cited as I), which is referred to as 'BLI'. This empirically inferred and generalizable inequality serves to bound from below the mean shear profile (pipe flow) or the mean vertical temperature gradient (convection at very large Rayleigh number). The purpose of this paper is to generalize BLI and test it in the much more complex double-diffusion experiment, in which there are two coupled mean fields which need to be bounded. The variational principle (II) will also be applied to this case (see the end of § 2).

Past work on double diffusion (Linden & Shirtcliffe 1978, hereinafter cited as LS), is reviewed first in § 2, and a slight modification of a boundary condition is made to avoid conflict with observations (Griffiths 1979) when another solute ('triple diffusion') is added. The weakest form of BLI is developed first (§ 3) so as to minimize the number of ad hoc *qualitative* assertions, and the stronger form of BLI appears in § 4. The weak form is then applied to the problem of triple diffusion, but comparison of the main result (5.12) with experiment cannot be made because measurements of only two of the relevant fluxes are available.

But for double diffusion the predicted bound on the salt/heat-flux ratio does agree with experiments and the bound is apparently realized in an asymptotic limit (of 'small' flux). This remarkable coincidence is interpreted as evidence favouring the validity of the variational principle which is summarized at the end of § 2.

2. Previous work and preliminary assumptions

Double-diffusive convection (Veronis 1968; Huppert & Moore 1976) can be realized by placing a deep layer of relatively cold, fresh (and light) water above another deep layer of hot, salty (and denser) water. Some time afterwards the vertical (z) variation of mean density $\bar{\rho}(z)$ is as shown in figure 1. The density inversion occurs because the diffusivity κ_T of the horizontally averaged thermal field $\bar{T}(z)$ exceeds the diffusivity κ_S of the salt field $\bar{S}(z)$. The $\bar{T}(z)$ field thereby tends to spread more in z than does the $\bar{S}(z)$ field, and this means that the unstably stratified $\bar{T}$ field will dominate in the determination of the density gradient $\bar{\rho}'(z)$ at $z \gg 0$ (or at $z \ll -h$). The gravitational instability of these regions initiates small-scale convection motions at the outer edges of the 'core', in which region the stabilizing salinity gradient is dominant. The small-scale and intermittent instabilities at the upper core edge ($z = 0$, as defined subsequently) rise like plumes into the convective layer (LS, figure 2), and this fluid is replaced by relatively homogeneous water sinking from above. This process removes the steadily diffusing heat and salt from the core, and also maintains the large-scale and energy-bearing motions in the convective layer. A statistically steady h is thereby established, provided that D is sufficiently large so that the temporal drift in ΔT, ΔS (figure 1) is negligible (LS).

The non-dimensional quantity T, in all that follows, denotes the product of the temperature with the (constant) thermal expansion coefficient, and the non-dimensional S denotes the contribution of the salinity to the equation of state. This is given by $\rho = S - T$, where ρ is the non-dimensional density. These densimetric units are also used for the convective fluxes, so that the *total* heat flux F_T and salt flux F_S, will have velocity units. By a proper identification of the S, T symbols the results which follow can be used in the (much more convenient) *isothermal* experiments, where two different substances (like sugar/salt) are used as the double-diffusers.

As indicated above, the equilibrium core thickness h is not an externally controllable parameter, but like (F_S, F_T) is dependent on $g\Delta S$, $g\Delta T$, κ_T, κ_S and the kinematic viscosity ν. Under certain conditions (most notably (2.10)) the relationship between F_T or F_S or h and $(g\Delta T, g\Delta S, \kappa_S, \kappa_T, \nu)$ is independent of $D \to \infty$ (figure 1) and independent of time. These qualitative aspects of the LS model are consistent with observations except when $R = \Delta T/\Delta S \to 1$. At this limit the system is on the verge of becoming top-heavy, and the interface becomes so chaotic that the picture of a diffusively balanced core must be modified.

For somewhat larger R attention is directed to the level $z = 0$ in figure 1, where the mean density $\bar{\rho}(z)$ equals the far-field density $\bar{\rho}(\infty)$. From now on, T, S and ρ are all measured relative to their far-field values, and the defining relation for $z = 0$ becomes

$$\bar{T}(0) = \bar{S}(0), \quad \text{or} \quad \int_0^\infty \bar{T}'(z)\,dz = \int_0^\infty \bar{S}'(z)\,dz, \tag{2.1a, b}$$

where the prime denotes a derivative in all that follows. The $z = 0$ level is physically significant because all of the overlying mean field is lighter than the far field ($z = \infty$), and conditional instability exists in the sense that the fluid at $z > 0$ is able to rise to infinity by means of its own buoyancy. On the other hand, the underlying fluid in the diffusive core is very stable, and figure 1 shows that

$$|\bar{S}'(0)| > |\bar{T}'(0)| \tag{2.2}$$

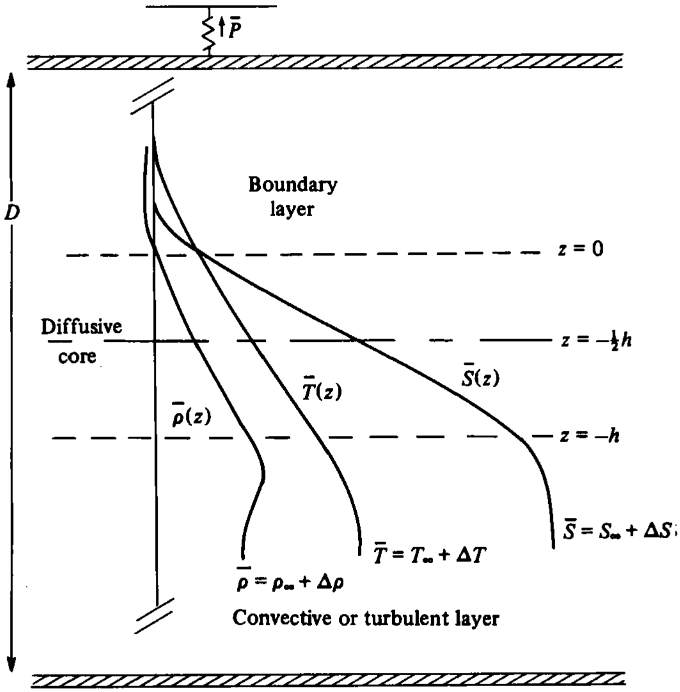

FIGURE 1. Schematic diagram of the horizontally averaged temperature $\bar{T}(z)$, salinity $\bar{S}(z)$, and density $\bar{\rho}(z)$ for two deep layers with given overall temperature and salinity differences (ΔT, ΔS). The two non-conducting boundaries are separated by a large distance D, so that the temporal 'run-down' of ΔT, ΔS is negligible. (The spring attached to the upper boundaries measures the statistically steady pressure $\bar{P}$.)

Because of this stability LS assume that both of the horizontally averaged convective fluxes vanish at $z = 0$ and in the underlying core. These (positive) convective fluxes are

$$\overline{w_0 T_0} = F_T + \kappa_T \bar{T}'(z), \quad \overline{w_0 S_0} = F_S + \kappa_S \bar{S}'(z), \tag{2.3a, b}$$

where $w_0(x, y, z, t)$ is the local turbulent vertical velocity, $T_0(x, y, z, t)$ is the deviation of the temperature from $\bar{T}$, and $S_0(x, y, z, t)$ is the salinity fluctuation. The total fluxes F_S, F_T are independent of z if $D = \infty$, and otherwise decrease linearly for $z \gg 0$ in the run-down experiments (LS).

Several reasons exist for modifying the (LS) assumption that $\overline{w_0 T_0} = 0 = \overline{w_0 S_0}$ at $z = 0$, one of these being a conflict with Griffiths' triple-diffusion experiment (§5). On physical grounds, moreover, one would expect an entrainment effect at $z = 0$ to be produced by the large eddies in the well-mixed layer. The motion produced by these at $z = 0$ should sweep up (entrain) some of the stable fluid at the core top, in much the same way as in the penetrative thermal convection experiment of Deardorff, Willis & Stockton (1980), where a reversal in sign of the buoyancy flux at the edge of the stable layer has been measured. By appealing to this related experiment, as well as to plausible physical grounds, we assume that $\overline{w_0 T_0} - \overline{w_0 S_0}$ changes sign near $z = 0$, i.e.

$$\overline{w_0 T_0} \leqslant \overline{w_0 S_0} \quad (-\tfrac{1}{2}h \leqslant z \leqslant 0), \tag{2.4}$$

and this qualitative assumption is testable. But the vertical average of $\overline{w_0 T_0} - \overline{w_0 S_0}$ must be positive, and also

$$F_S < F_T \tag{2.5}$$

is required by the global mechanical-energy integral.

All terms are positive in equations (2.4)–(2.5), and thus the product of these inequalities yields

$$F_S(F_T + \kappa_T \overline{T}') \leqslant F_T(F_S + \kappa_S \overline{S}') \quad (-\tfrac{1}{2}h \leqslant z \leqslant 0),$$

where (2.3*a*, *b*) have been used. Since $\overline{T}'$ and $\overline{S}'$ are both negative, we then obtain

$$\frac{F_S}{F_T} \geqslant \frac{\kappa_S \overline{S}'(0)}{\kappa_T \overline{T}'(0)}. \tag{2.6}$$

Equation (2.2) already implies a lower bound

$$r = \frac{F_S}{F_T} > \frac{\kappa_S}{\kappa_T}, \tag{2.7}$$

for this flux ratio, and a sharper bound will be found (§3) by introducing another dynamical constraint. From (2.4)–(2.5) we also obtain

$$\int_{-\frac{1}{2}h}^{0} dz \frac{\overline{w_0 T_0}}{F_T} \leqslant \int_{-\frac{1}{2}h}^{0} dz \frac{\overline{w_0 S_0}}{F_S}$$

and (2.3*a*, *b*) then yield

$$\frac{\kappa_T}{F_T}(\overline{T}(0) - \tfrac{1}{2}\Delta T) \leqslant \frac{\kappa_S}{F_S}(\overline{S}(0) - \tfrac{1}{2}\Delta S),$$

where $\frac{1}{2}\Delta T$, $\frac{1}{2}\Delta S$ are values of the mean $(\overline{T}, \overline{S})$ fields at the level of symmetry (figure 1). With the help of (2.1*a*) this inequality becomes

$$\overline{T}(0) \leqslant \tfrac{1}{2}\Delta T \frac{r - \tau R}{r - \tau}, \tag{2.8}$$

$$\tau \equiv \frac{\kappa_S}{\kappa_T}. \tag{2.9}$$

Since all temperatures are positive by convention, the right-hand side of (2.8) must be positive. Using (2.7) and (2.5) we then get

$$R \leqslant \frac{r}{\tau} \leqslant \frac{1}{\tau}. \tag{2.10}$$

The first two terms in this inequality give the LS *cutoff point*:

$$R = \frac{r}{\tau}, \tag{2.11}$$

at which $\overline{T}(0)$ vanishes in (2.8), and vanishing F_T is demanded on physical grounds. In this limit the equilibrium core thickness must approach infinity as $h \simeq \kappa_T \Delta T / F_T$.

The basis for sharpening (2.7) will be the well known 'power integrals' for the production–dissipation of kinetic energy, temperature variance and salinity variance in the entire fluid (figure 1). If this is bounded by rigid non-conducting surfaces at

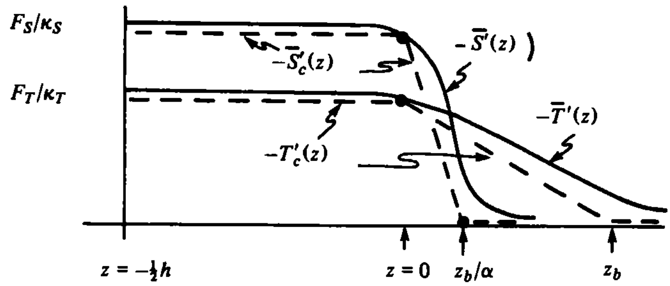

FIGURE 2. The gradients for the upper half of figure 1 are sketched in the solid curves, with F_S/κ_S and F_T/κ_T being the respective values of the salinity, and temperature gradients at the centre of the diffusive core. The broken-line trial functions $(-\bar{T}_c'(z), -\bar{S}_c'(z))$ are hinged at the $z = 0$ datum level (see text), and z_b is the thickness of the thermal boundary layer.

$z \simeq \pm\infty$, and, if the time derivatives of the variances are negligible, then these power integrals are

$$\overline{\overline{gw_0(T_0 - S_0)}} = \nu\overline{\overline{(\nabla\times\mathbf{V}_0)^2}}, \tag{2.12}$$

$$-\overline{\overline{w_0 T_0 \bar{T}'(z)}} = \kappa_T\overline{\overline{(\nabla T_0)^2}}, \tag{2.13}$$

$$-\overline{\overline{w_0 S_0 \bar{S}'(z)}} = \kappa_S\overline{\overline{(\nabla S_0)^2}}, \tag{2.14}$$

where $\mathbf{V}_0 = \{u_0, v_0, w_0\}$ denotes the non-divergent velocity and its $\{x, y, z\}$ components, and the second bar denotes a z-average. Another useful power integral is obtained by multiplying the field equation for $\partial T_0(x,y,z,t)/\partial t$ with $S_0(x,y,z,t)$, by multiplying (Fick's) equation for $\partial S_0/\partial t$ with T_0, and by adding and averaging the resulting equation for $\partial(T_0 S_0)/\partial t$ to obtain (Stern 1975, p. 206)

$$-\overline{\overline{w_0 T_0 \bar{S}'}} - \overline{\overline{w_0 S_0 \bar{T}'}} = (\kappa_S + \kappa_T)\,\overline{\overline{\nabla T_0 \,.\, \nabla S_0}}. \tag{2.15}$$

If averages are taken only over a finite vertical interval, such as the boundary layer in figure 1, one must add energy-export terms to (2.12)–(2.15). In § 3 we focus these global equations on the boundary layer by means of further assumptions. The reader may prefer to pass directly to this section, without interrupting the argument with the following recapitulation of the variational principle (II). The latter is only used to interpret the result of BLI, and is therefore somewhat independent.

For the classical problem of thermal convection (II) between two rigid, conducting, horizontal boundaries our general thermodynamic-selection principle implies that the mean pressure $\bar{P}$ acting at one boundary is an extremum (maximum or minimum) provided that the heat flux as well as the temperature difference between the boundaries is held constant in the variation with respect to the degenerate class of solutions. The significance of these three parameters is that they measure the 'state of the reservoir', $\bar{P}$ being the momentum transport to the latter. The general application of the theory requires an identification of all the reservoir state parameters, and their inclusion as constraints in the variational problem. Some of these constraints may be ignored (or discarded) to obtain an approximate variational solution.

In the run-down experiment to which figure 2 refers, the deep ($D \to \infty$) mixed layers are identified as the reservoirs, whose temperature and salinity ($\Delta T, \Delta S$) we are free to fix. Moreover, the slow ($D \to \infty$) temporal decrease of $\Delta T, \Delta S$ is also identified as a reservoir state parameter since the interfacial fluxes (F_S, F_T) are in fact determined from the measurement of these decreases. The boundary pressure $\bar{P}$, which could be measured by the deflexion of the spring in figure 1, is obviously a reservoir state parameter as in II. Are there other higher-order statistical parameters which should be counted as reservoir state parameters, and included as constraints in the extremization of $\bar{P}$ with ΔT, ΔS, F_S, F_T held constant? If so, then we agree to discard them in the sense of a variational approximation.

Now F_S, F_T depend explicitly on the structure of $\bar{T}'(z), \bar{S}'(z)$ whereas the momentum transport $\bar{P}$ has an explicit dependence on $\overline{\overline{w_0^2}}$ (II) or on the properties of the large eddies in the mixed layer. We therefore propose to manipulate the variational principle and to discard constraints, so as to focus the result on the boundary layer exclusively. Extremization of $\bar{P}$ with F_S, F_T held constant is equivalent to extremization of F_S with F_T, $\bar{P}$ held constant. We now discard the $\bar{P}$ constraint and merely seek an extremum of F_S with F_T (and $\Delta T, \Delta S$) held constant or, equivalently, an extremum of $r = F_S/F_T$ with a suitably defined non-dimensional F_T held constant. The only dynamical constraint used in this first-order variational problem is the *inequality* in § 3. It is rather obvious, therefore, that the solution of the variational problem will lie on the bounding surface of the inequality in the function space to which it applies. At this level of approximation the variational principle merely converts a dynamical inequality into an equality.

3. The inequality pertaining to the energetics of the boundary layer

The BLI constructed for pipe flow and thermal convection (I) was based on the plausible idea that the mean field gradients in the viscous conductive boundary layers are sufficiently large so that some perturbation is capable of releasing more energy than it dissipates in the *boundary layer* (suitably defined). The quantification of this idea involves a truncation of the power integral at the edge of the boundary layer, and the replacement of the fluctuating fields by 'test perturbations' which scan the permitted function space. The calculation thereby determines bounds on the mean gradients such that the power integral can be satisfied. The formalism is essentially a means of defining and bounding the boundary-layer Reynolds or Raleigh number in shear flow or thermal turbulence. The following generalization will do essentially the same thing for double diffusion, except that there are at least two Rayleigh numbers for this problem as well as a Prandtl-number and Schmidt-number dependence. Thus dimensional reasoning is of no avail, and a real challenge is presented to our formalism.

The first step in the construction involves the separation of each of the realized profiles $\{\bar{T}'(z), \bar{S}'(z)\}$ into two parts, the simple parts being denoted by $\{\bar{T}'_{c0}(z), \bar{S}'_{c0}(z)\}$ and the other parts being the respective residuals. Each of the two simple parts consists of three broken straight lines (figure 2), one segment of which is drawn tangent to the corresponding *realized* profile from the point at $z = 0$ on each of these profiles. Thus, the simple curves satisfy $\bar{T}'_{c0}(0) = \bar{T}'(0)$, $\bar{S}'_{c0}(0) = \bar{S}'(0)$, and the simple curves bound the real curves from below by virtue of this tangent construction. The simple curves are broken straight lines like the $\bar{T}'_c(z), \bar{S}'_c(z)$ in figure 2, except that the latter are not necessarily tangent to the real profiles. The $(\bar{T}'_c, \bar{S}'_c)$ are a more general class of curves (all of which are 'hinged' at $z = 0$ on the real profile) in which the curves $(\bar{T}'_{c0}, \bar{S}'_{c0})$ will be embedded. The realized thermal boundary-layer thickness z_{b0} is defined by the tangent intercept, and is slightly larger than the z_b value indicated in figure 2.

It would be nice to have an equally unambiguous way of dividing up the real turbulent fluctuations $(\mathbf{V}_0, T_0, S_0)$, such that one of its parts $(\mathbf{V}_{c0}, \theta_{c0}, \Sigma_{c0})$ could be associated with the boundary layer, and with $\{\bar{T}'_{c0}(z), \bar{S}'_{c0}(z)\}$. Because the boundary-layer $(0 \leqslant z \leqslant z_{b0})$ instabilities export energy to the convective layer $z > z_{b0}$ we would then be inclined to assert that the 'production' of velocity, temperature, and salinity variance (*cf.* (2.12)–(2.14)) as defined respectively by

$$\int_0^{z_{b0}} dz \begin{Bmatrix} g\overline{w_{c0}(\theta_{c0}-\Sigma_{c0})} \\ -\overline{w_{c0}\theta_{c0}}\,\bar{T}'_{c0} \\ -\overline{w_{c0}\Sigma_{c0}}\,\bar{S}'_{c0} \end{Bmatrix} \tag{3.0a}$$

exceeds the corresponding 'dissipation' term in:

$$\int_0^{z_{b0}} dz \begin{Bmatrix} \nu\overline{(\nabla \times \mathbf{V}_{c0})^2} \\ \kappa_T\overline{(\nabla\theta_{c0})^2} \\ \kappa_S\overline{(\nabla\Sigma_{c0})^2} \end{Bmatrix}. \tag{3.0b}$$

Since we do not know how to make such a division of the turbulence, the idea will be quantified by asserting that some function $(\mathbf{V}_{c0}, \theta_{c0}, \Sigma_{c0})$ exists for which (3.0*a*) exceeds (3.0*b*). The class of permissible functions are non-divergent and satisfy side conditions at $z = 0, z_{b0}$ as indicated below. For given $(-\bar{T}'(0), -\bar{S}'(0))$, and for $z_{b0} \to 0$, there exist *no* permissible functions $\mathbf{V}_{c0}$, θ_{c0}, Σ_{c0} such that (3.0*a*) exceeds (3.0*b*), and therefore our assertion will set a lower bound on the realizable z_{b0}. We showed (I) that the same assertion was both correct and non-trivial for the shear and thermal turbulence problem, and, if the same thing can be done for the present problem, then we may claim that BLI 'captures' a general property of turbulent boundary layers.

Since our assertion is a definite inequality, it must be equally valid if $(-\bar{T}'_{c0}(z), -\bar{S}'_{c0}(z))$ in (3.0*a*) is replaced by somewhat *smaller* values $(-\bar{T}'_c(z), -\bar{S}'_c)$ in the more general class of hinged profiles, *provided* the differences $(\bar{T}'_{c0}-\bar{T}'_c, \bar{S}'_{c0}-\bar{S}'_c)$ are sufficiently small. This observation allows us to embed the measurable fields $(\bar{T}'_{c0}, \bar{S}_{c0})$ in the more general class.

Furthermore, the definite inequality with respect to the $(\mathbf{V}_{c0}, \theta_{c0}, \Sigma_{c0})$ functions implies the existence of other permissible functions $\mathbf{V}_c, \theta_c, \Sigma_c$ for which production *equals* dissipation, the reason being that the relative importance of dissipation can always be made to increase by merely decreasing the horizontal wavelength of $(\mathbf{V}_{c0}, \theta_{c0}, \Sigma_{c0})$ (I). This observation allows us to write BLI in the more convenient form of an equality containing 'dummy' variables like $(\mathbf{V}_c, \theta_c, \Sigma_c)$. By permitting these to scan the whole permissible range we thereby obtain inequalities or bounds for real parameters like $\bar{T}'(0), \bar{S}'(0), z_{b0}, F_S, F_T$. The physical content of this elaborate construction is that the real $\{-\bar{T}'(z), -\bar{S}'(z)\}$ can be (respectively) bounded below by some member $\{-\bar{T}'_c(z), -\bar{S}'_c(z)\}$ of the class of hinged profiles, each of which is capable of satisfying the truncated power integral for some $\mathbf{V}_c, \theta_c, \Sigma_c$.

The BLI conjecture is testable, and if correct then the same inequality must hold for a large class of *neighbouring* (degenerate) solutions of the field equations, these neighbours having somewhat different values of $\bar{T}'(z), \bar{S}'(z), F_T, F_S$ than the realized solution. Thus BLI provides a constraint on the class of degenerate dynamical solutions to which we may apply our variational theory. Accordingly (§2), we minimize $r = F_S/F_T$ for a given F_T (and for given ΔT, ΔS, κ_T, κ_S, ν), and subject to BLI. Of particular interest is the minimum r at the LS cutoff point where $F_T \to 0$.

A variational theory of turbulence also allows us to obtain interesting quantitative approximations by 'borrowing' observed qualitative features, and also by taking certain liberties with trial-function representations. For example, the boundary conditions at $z = (0, z_b)$ on $(\mathbf{V}_c, \theta_c, \Sigma_c)$ need not be exact, but they should be 'passive' and incorporate the idea that the unstable perturbations are not driven by fluxes from outside the boundary layer. These boundary conditions on $\mathbf{V}_c, \theta_c, \Sigma_c$ should also incorporate the idea that the vertical convective fluxes at $z = 0$ are small compared to those at $z = z_b$, where energy is exported to greater heights. The significance of these side conditions will appear in the following restatement of BLI.

The solid curves in figure 2 are sketches of dynamically possible temperature and salinity gradients. The trial functions $\{-\overline{T}'_c(z), -\overline{S}'_c(z)\}$ mentioned previously, are the broken straight lines 'hinged' at $z = 0$ on the corresponding $(-\overline{T}'(z), -\overline{S}'(z))$ profiles. The trial functions are also required to satisfy the defining relations (2.1*b*), and thus they are given by

$$-\overline{T}'_c(z) = -\overline{T}'(0) \times \begin{cases} 1 & (z < 0), \\ (1 - z/z_b) & (0 < z < z_b), \\ 0 & (z > z_b); \end{cases} \tag{3.1}$$

$$-\overline{S}'_c(z) = -\overline{S}'(0) \times \begin{cases} 1 & (z < 0), \\ (1 - \alpha z/z_b) & (0 < z < z_b/\alpha), \\ 0 & (z > z_b/\alpha); \end{cases} \tag{3.2}$$

$$\int_0^\infty \overline{T}'_c(z)\, dz = \int_0^\infty \overline{S}'_c(z)\, dz; \tag{3.3}$$

where z_b, α are free parameters. When (3.1)–(3.2) are substituted in (3.3), and when (2.2) is noted, we get

$$\alpha = \frac{\overline{S}'(0)}{\overline{T}'(0)} > 1. \tag{3.4}$$

The inequality (2.6), on the other hand, implies

$$\alpha \leqslant \frac{\kappa_T}{\kappa_S} \frac{F_S}{F_T}. \tag{3.5}$$

The trial functions $\mathbf{V}_c, \theta_c, \Sigma_c$ release as much energy in $0 \leqslant z \leqslant z_b$ as they dissipate, which means that they satisfy the truncated power integrals (2.12)–(2.15), or

$$g \int_0^{z_b} \overline{w_c(\theta_c - \Sigma_c)}\, dz = \nu \int_0^{z_b} \overline{(\nabla \times \mathbf{V}_c)^2}\, dz, \tag{3.6}$$

$$-\int_0^{z_b} \overline{w_c \theta_c} \overline{T}'_c\, dz = \kappa_T \int_0^{z_b} \overline{(\nabla \theta_c)^2}\, dz, \tag{3.7}$$

$$-\int_0^{z_b} \overline{w_c \Sigma_c} \overline{S}'_c\, dz = \kappa_S \int_0^{z_b} \overline{(\nabla \Sigma_c)^2}\, dz, \tag{3.8}$$

$$-\int_0^{z_b} (\overline{w_c \theta_c}\, \overline{S}'_c + \overline{w_c \Sigma_c}\, \overline{T}'_c)\, dz = (\kappa_S + \kappa_T) \int_0^{z_b} (\overline{\nabla \theta_c \cdot \nabla \Sigma_c})\, dz, \tag{3.9}$$

as well as the continuity equation

$$0 = \nabla \cdot \mathbf{V}_c = \frac{\partial u_c}{\partial x} + \frac{\partial v_c}{\partial y} + \frac{\partial w_c}{\partial z}$$

and certain

$$\text{'side conditions'} \tag{3.10}$$

to be mentioned subsequently. The content of this construction emerges with the assertion that any dynamically possible $(-\bar{T}'(z), -\bar{S}'(z))$, including the realized one, can be bounded below by some $(-\bar{T}'_c(z), -\bar{S}'_c(z))$ for which (3.6)–(3.10) can be satisfied [using an allowable $(\mathbf{V}_c, \theta_c, \Sigma_c)$]. The main assertion implies

$$\bar{T}(0) = -\int_0^\infty \bar{T}'(z)\,dz \geqslant -\int_0^\infty \bar{T}'_c(z)\,dz = -\tfrac{1}{2}z_b\bar{T}'(0), \tag{3.11a}$$

$$\bar{S}(0) = -\int_0^\infty \bar{S}'(z)\,dz \geqslant -\int_0^\infty \bar{S}'_c(z)\,dz = -\tfrac{1}{2}z_b\bar{T}'(0) \tag{3.11b}$$

for some allowable z_b. By using (3.11*a*) in (2.8) we get

$$\frac{\Delta T(r-\tau R)}{r-\tau} \geqslant -\bar{T}'(0)\min z_b, \tag{3.12}$$

where $\min z_b$ is the smallest value of z_b with respect to *all* permissible test functions (and with $\bar{T}'(0), \bar{S}'(0)$ held constant). The evaluation of $\min z_b$ as a function of $\bar{T}'(0)$, $\bar{S}'(0)$ and the use of (2.6), will then give an inequality which all dynamically possible solutions (having different F_S, F_T) must satisfy. The main variational principle (2.11) is then applied to this constraining inequality.

Some kind of boundary condition (3.10) for $(\mathbf{V}_c, \theta_c, \Sigma_c)$ at $z = 0$ and $z = z_b$ is necessary, for otherwise the minimum in (3.12) will be zero and the construction will be vacuous. The specification of this poses a difficulty, and the simplest resolution would be to assume some simple form like: $\sin(\pi z/2z_b)\sin kx \sin ly$ for $(w_c, \theta_c, \Sigma_c)$. This form will satisfy

$$\overline{w_c\theta_c} \equiv \Gamma(z)\begin{cases} = 0 & (z = 0), \\ > 0 & (0 < z < z_b); \end{cases} \tag{3.13}$$

$$\frac{d\Gamma(0)}{dz} = 0; \tag{3.14}$$

$$\frac{d^2\Gamma}{dz^2} \text{ is maximal at } z = 0; \tag{3.15}$$

$$\Sigma_c = f\theta_c; \tag{3.16}$$

where f is a free parameter. Equations (3.13)–(3.16) obviously include test functions that are far less restrictive and arbitrary than the sinusoidal ones, and thus we propose:

$$\text{Equations (3.13)–(3.16) substituted in condition (3.10).} \tag{3.17}$$

This constitutes the *weak* form of BLI, and a stronger form (4.1)–(4.2) will be introduced in §4. Note that there is no inconsistency between (3.13), (3.14) and the finite flux value implied by (2.4), because the latter is attributed to the large-scale eddies while the former only describes the smaller-scale instabilities in the boundary layer. On physical grounds these should have rather highly correlated temperature–salinity fluctuations, so that (3.16) is a reasonable trial approximation. Moreover, (3.16) greatly simplifies the calculation by making (3.9) redundant if (3.7) and (3.8) are satisfied. By dividing (3.7) into (3.8), and by using (3.1), (3.2), (3.13), (3.16) one obtains

$$\frac{\bar{S}'(0)}{f\kappa_S}\int_0^{z_b/\alpha}\left(1-\frac{\alpha z_b}{z_b}\right)\Gamma(z)\,dz = \frac{\bar{T}'(0)}{\kappa_T}\int_0^{z_b}(1-z/z_b)\,\Gamma(z)\,dz. \tag{3.18}$$

Equations (3.7) and (3.8) also imply $f > 0$, (3.6) implies $1-f > 0$, and therefore

$$0 < f \leqslant 1. \tag{3.19}$$

By introducing the new function

$$\Gamma_0(\zeta) \equiv \Gamma(\zeta z_b)$$

equation (3.18) transforms to

$$\frac{\bar{S}'(0)}{\bar{T}'(0)} = \frac{f\alpha\kappa_S}{\kappa_T}\frac{\int_0^1 \Gamma_0(\zeta)(1-\zeta)\,d\zeta}{\int_0^1 \Gamma_0(\zeta/\alpha)(1-\zeta)\,d\zeta}. \tag{3.20}$$

The function $\Gamma_0(\zeta)$ must satisfy the same side relations as $\Gamma(z)$ in (3.13)–(3.15). Thus $\Gamma_0''(\zeta)$ is a non-increasing function of ζ, with $\Gamma_0(0) = 0 = \Gamma_0'(0)$, and consequently

$$\Gamma_0(\zeta) = \int_0^\zeta dx\,\Gamma_0'(x) = \int_0^\zeta dx\int_0^x \Gamma_0''(y)\,dy \geqslant \Gamma_0''(\zeta)\int_0^\zeta dx\int_0^x dy = \tfrac{1}{2}\zeta^2\Gamma''(\zeta),$$

$$\Gamma_0(\zeta) \leqslant \Gamma_0''(0)\int_0^\zeta dx\int_0^x dy = \tfrac{1}{2}\zeta^2\Gamma_0''(0).$$

Now consider the function

$$\Phi(\zeta,\eta) \equiv \Gamma_0(\zeta) - \frac{\Gamma_0(\eta)\,\zeta^2}{\eta^2}, \tag{3.21}$$

which has the property

$$\Phi_{\zeta\zeta} = \Gamma_0''(\zeta) - \frac{2\Gamma_0(\eta)}{\eta^2} \leqslant \frac{2\Gamma_0(\zeta)}{\zeta^2} - \frac{2\Gamma_0(\eta)}{\eta^2} = \frac{2\Phi}{\zeta^2}.$$

This implies that if $\Phi < 0$ in any region then $\Phi_{\zeta\zeta} < 0$, and $\Phi\Phi_{\zeta\zeta}$ is positive. However, $\Phi = 0$ at $\zeta = 0$ and also at $\zeta = \eta$, and therefore Φ must be non-negative at all intervening ζ; for otherwise Φ would be negative everywhere between two of its zeroes, $\Phi\Phi_{\zeta\zeta}$ would be positive, and the identity

$$\int d\zeta\frac{\partial}{\partial\zeta}\Phi\Phi_\zeta = \int dz\,(\Phi\Phi_{\zeta\zeta}+\Phi_\zeta^2) > 0$$

would be contradicted when the limits of integration are taken as these two zeroes. This proves that $\Phi(\zeta) \geqslant 0$ for $0 \leqslant \zeta \leqslant \eta$, and (3.21) gives

$$\Gamma_0(\zeta) \geqslant \Gamma_0(\eta)\,\zeta^2/\eta^2 \quad (0 \leqslant \zeta \leqslant \eta)$$

or

$$\Gamma_0(\xi\eta) \geqslant \xi^2\Gamma_0(\eta) \quad (0 \leqslant x \leqslant 1).$$

By setting $\xi = 1/\alpha$ (equation (3.4)), and by setting η equal to ζ we get

$$\Gamma_0(\zeta/\alpha) \geqslant \frac{\Gamma_0(\zeta)}{\alpha^2} \quad (\alpha = \bar{S}'(0)/\bar{T}'(0)). \tag{3.22}$$

Equation (3.20) now becomes

$$\frac{\bar{S}'(0)}{\bar{T}'(0)} \leqslant f\frac{\kappa_S}{\kappa_T}\alpha^3, \tag{3.23}$$

and (3.4) and (3.19) then give

$$\frac{\bar{S}'(0)}{\bar{T}'(0)} \geqslant \frac{(\kappa_T/\kappa_S)^{\frac{1}{2}}}{f^{\frac{1}{2}}} \geqslant \left(\frac{\kappa_T}{\kappa_S}\right)^{\frac{1}{2}}. \tag{3.24}$$

The final result obtained with the help of (2.6) is that

$$\frac{F_S}{F_T} \geqslant \left(\frac{\kappa_S}{\kappa_T}\right)^{\frac{1}{2}}, \tag{3.25}$$

and thus we see that all dynamically possible solutions of the equations of motion (including the realized one) must have a flux ratio in excess of

$$\frac{F_S}{F_T} = \left(\frac{\kappa_S}{\kappa_T}\right)^{\frac{1}{2}}. \tag{3.26}$$

This bound can only be reached if $f \to 1$ or $\overline{w_c\theta_c} - \overline{w_c\Sigma_c} \to 0$, which suggests that a small non-dimensional buoyancy flux is a necessary condition for the realization of (3.26). Further support of this conjecture is given in § 4, but one cannot be sure that the minimum salt/heat-flux ratio will not be raised when additional constraints are added.

4. Stronger constraints

The weak side conditions (3.13)–(3.15) will now be replaced by the stronger boundary conditions

$$0 = \frac{\partial w_c}{\partial z} = \frac{\partial \theta_c}{\partial z} = u_c = v_c \quad (z = z_b), \tag{4.1}$$

$$0 = w_c = \theta_c = \frac{\partial^2 w_c}{\partial z^2} \quad (z = 0). \tag{4.2}$$

The maximum (w_c, θ_c) amplitudes imposed here at the $z = z_b$ boundary are in accord with our picture of an intermittent instability exchanging fluid with the convective layer $(z > z_b)$, and the same boundary condition was used for the test function in our earlier work. The 'slippery' (or Rayleigh free boundary) condition (4.2) imposed at the entraining boundary is arbitrary, but a 'rigid' boundary condition at $z = 0$ will just increase the numerical coefficient at the end of (4.8). Equation (3.16) is again used to simplify the calculation, to make (3.9) redundant, and to retain the important integral (3.18).

The latter equation and (3.6) and (3.7) are made non-dimensional by using z_b as the length scale, and also by using

$$\mathbf{V}_1 \equiv \frac{z_b}{\kappa_T}\mathbf{V}_c, \quad T_1 = \frac{A^{-1}\theta_c}{z_b(-\bar{T}'(0))},$$

where A^{-1} is a non-dimensional constant. The three integral equations in question then become

$$\left.\begin{aligned} f &= \frac{\alpha}{\tau}\frac{\int_0^{\alpha^{-1}}(1-\zeta\alpha)\,\overline{w_1T_1}\,d\zeta}{\int_0^1(1-\zeta)\,\overline{w_1T_1}\,d\zeta} \\ \int_0^1 \overline{w_1T_1}(1-\zeta)\,d\zeta &= A\int_0^1 \overline{(\nabla T_1)^2}\,d\zeta, \\ \frac{(1-f)\,z_b^4 g[-\overline{T}'(0)]}{\nu\kappa_T} &= \frac{\int_0^1 \overline{(\nabla\times \mathbf{V}_1)^2}\,d\zeta}{A\int_0^1 \overline{w_1T_1}\,d\zeta}. \end{aligned}\right\} \tag{4.3}$$

By eliminating A, and by using (3.12) to eliminate z_b, we get

$$\left(\frac{r-\tau R}{r-\tau}\right)^4 \frac{(g\Delta T)^4}{\nu\kappa_T[-g\overline{T}'(0)]^3} \geqslant \min\frac{I}{1-f}, \tag{4.4}$$

where

$$I \equiv \left\{\frac{\int_0^1 \overline{(\nabla T_1)^2}\,d\zeta \int_0^1 \overline{(\nabla\times\mathbf{V}_1)^2}\,d\zeta}{\left[\int_0^1 \overline{w_1T_1}\,d\zeta\right]^2}\right\}\left\{\frac{\int_0^1 \overline{w_1T_1}\,d\zeta}{\int_0^1 \overline{w_1T_1}(1-\zeta)\,d\zeta}\right\}, \tag{4.5}$$

and where the minimization in (4.4) is with respect to $\mathbf{V}_1, T_1$.

Variations of $I/(1-f)$ with respect to T_1, give an Euler equation in which $\nabla^2 T_1$ is proportional to w_1. Since (w_1, T_1) both vanish at $\zeta = 0$ (4.2) it follows that $\partial^2 T_1/\partial\zeta^2$ also vanishes at $\zeta = 0$. Equation (4.2) then implies that the third derivative of $\overline{w_1T_1}$ vanishes, or $\partial^2\overline{w_c\theta_c}/\partial z^2$ has an extremum at $z = 0$. This extremum is tentatively assumed (and subsequently verified from the eigenfunctions) to be a *maximum*, so that (3.15) is still satisfied, and the inequality (3.23) still holds, i.e.

$$f \geqslant \frac{1}{\tau}\left(\frac{\overline{T}'(0)}{\overline{S}'(0)}\right)^2 \geqslant \frac{\tau}{r^2}, \tag{4.6}$$

where (2.6) has been used in the last part of (4.6).

The inequality (4.4) will therefore be preserved if f is replaced by its lower bound (4.6), and if $-\overline{T}'(0)$ (in (4.4)) is replaced by the lower bound

$$-\kappa_T\overline{T}'(0) = F_T - (\overline{w_0T_0})_{z=0} > F_T - (\overline{w_0S_0})_{z=0} > F_T - F_S,$$

where (2.4) has been used. Equation (4.4) then becomes

$$\frac{\tau}{r^2} \leqslant 1 - \left(\frac{r-\tau}{r-\tau R}\right)^4 \left[\frac{g(F_T-F_S)\,\nu^{\frac{1}{3}}}{(g\Delta T)^{\frac{4}{3}}\,\kappa_T^{\frac{2}{3}}}\right]^3 \min I, \tag{4.7}$$

and it only remains to compute the purely numerical value of $\min I$.

The first of the two terms on the right-hand side of (4.5) has, in view of the boundary conditions, a minimum value equal to the critical Rayleigh number for the onset of pure thermal convection between two slippery boundaries located at $\zeta = 0$ and $\zeta = 2$ ($\zeta = 1$ is the plane of symmetry), and thus

$$\frac{\int_0^1 \overline{(\nabla T_1)^2}\, d\zeta \int_0^1 \overline{(\nabla \times \mathbf{V}_1)^2}\, d\zeta}{\left(\int_0^1 \overline{w_1 T_1}\, d\zeta\right)^2} \geqslant \frac{\frac{27}{4}\pi^4}{2^4}.$$

The minimum value of (4.5) is certainly greater than this because the last term in (4.5) is certainly greater than unity. Therefore the inequality is preserved if $\min I$ is replaced by $27\pi^4/4(2^4)$.

A sharper bound can be obtained (if desired) by tentatively assuming and subsequently verifying that the eigenfunction which minimizes (4.5) is such that $\overline{w_1 T_1}$ is a monotonic increasing function of $\zeta \leqslant 1$, in which case we have the identity (I)

$$\frac{\int_0^1 \overline{w_1 T_1}\, d\zeta}{\int_0^1 \overline{w_1 T_1}(1-\zeta)\, d\zeta} \geqslant \frac{1}{\int_0^1 (1-\zeta)\, d\zeta} = 2,$$

and therefore

$$I \geqslant \tfrac{27}{28}\pi^4.$$

The use of this in (4.7) then gives

$$1 - \frac{\tau}{r^2} \geqslant \left(\frac{r-\tau}{r-\tau R}\right)^4 \left[\frac{g(F_T - F_S)\, \nu^{\frac{1}{3}}}{(g\Delta T)^{\frac{4}{3}}\, \kappa_T^{\frac{2}{3}}}\right]^3 \times \tfrac{27}{32}\pi^4. \tag{4.8}$$

For given $\Delta T, \Delta S$ all dynamically possible solutions, including the realized one, must have F_S, F_T values which satisfy the inequality (4.8). Our variational principle requires the *realized* values of F_S, F_T to lie approximately on the boundary surface of the inequality, which is given by

$$1 - \frac{\tau}{r^2} = \left(\frac{r-\tau}{r-\tau R}\right)^4 \left[\frac{g(F_T - F_S)\, \nu^{\frac{1}{3}}}{(g\Delta T)^{\frac{4}{3}}\ \ \kappa_T^{\frac{2}{3}}}\right]^3 \times \tfrac{27}{32}\pi^4. \tag{4.8a}$$

One of the main conclusions from this is that the asymptotic result (3.26), or $r = \tau^{\frac{1}{2}}$ holds only if the non-dimensional buoyancy flux on the right-hand side of (4.8*a*) is small, and some experimental numbers will be given in § 6.

Since the flux ratio must be $r \leqslant 1$, and since the density ratio must be $R \geqslant 1$, (4.8*a*) also implies

$$\frac{g(F_T - F_S)}{(g\Delta T)^{\frac{4}{3}}} \left(\frac{\nu}{\kappa_T^2}\right)^{\frac{1}{3}} \leqslant (1-\tau)^{\frac{1}{3}} \left(\frac{32}{27\pi^4}\right)^{\frac{1}{3}}. \tag{4.9}$$

This bound for the buoyancy flux should be relevant when $R \to 1$.

5. Multiple diffusion

A straightforward generalization of (3.25) and (3.26) will be obtained when another solute is added to the bottom layer in figure 1, with ΔX denoting its excess non-dimensional concentration relative to the upper layer, and with $d\overline{X}/dz < 0$ denoting the horizontally averaged concentration in the statistically steady state. Both the diffusivity κ_X of this substance and κ_S are to be smaller than κ_T, i.e.

$$\kappa_X < \kappa_T, \quad \kappa_S < \kappa_T. \tag{5.1}$$

The concentration of X must now be added to our previous expressions for non-dimensional density, and thus the $z = 0$ datum level (where $\bar{\rho}(0) = 0$) is now given by

$$-\int_0^\infty \frac{d\bar{T}}{dz}dz = -\int_0^\infty \frac{d\bar{S}}{dz}dz - \int_0^\infty \frac{d\bar{X}}{dz}dz. \tag{5.2}$$

The trial functions $(-\bar{T}'_c, -\bar{S}'_c, -\bar{X}'_c)$ are again 'hinged' at $z = 0$, with the temperature–salinity profiles given by (3.1) and (3.2), and with the new profile given by

$$-\frac{d\bar{X}_c}{dz} = -\bar{X}'(0)\begin{cases} 1 & (z < 0) \\ 1 - \alpha_X z/z_b & (0 < z < z_b/\alpha_X), \\ 0 & (z > z_b/\alpha_X), \end{cases} \tag{5.3}$$

where α_X (like α) is a free parameter which measures the thickness of the solute boundary layer relative to the thermal boundary layer. Because of (5.1) the relative boundary layers will be constrained by

$$\alpha_X > 1 \quad (\alpha > 1). \tag{5.4}$$

Since the trial profiles are required to satisfy (5.2), the substitution of (5.3), (3.1), (3.2) into this relation gives

$$-\bar{T}'(0) = -\frac{\bar{S}'(0)}{\alpha} - \frac{\bar{X}'(0)}{\alpha_X}. \tag{5.5}$$

The trial perturbation X_c for the X-fluctuation, like the salinity trial function Σ_c (3.16), is taken to be perfectly correlated with θ_c, i.e.

$$X_c = f_X\theta_c, \quad \Sigma_c = f\theta_c, \tag{5.6}$$

so as to render the power integrals for the production of S, T correlation, S, X correlation, and X, T correlation redundant when the three power integrals for T, S, X variances are satisfied. The latter three equations imply that the free parameters (f_X, f) are both positive. The generalized mechanical energy equation (3.6) containing the buoyancy term $gw_c(\theta_c - \Sigma_c - X_c)$ implies that $1 - f - f_X$ must be positive, and therefore

$$0 \leqslant f_X \leqslant 1 - f \leqslant 1. \tag{5.7}$$

Equations (3.7) and (3.8) are unaltered, equations (3.13)–(3.15) are unaltered, and therefore (3.23) is unaltered. A completely analogous relation for $\bar{X}'(0)/\bar{T}'(0)$ is obtained when the T–X variance equations are combined, *viz.*

$$\frac{\bar{X}'(0)}{\bar{T}'(0)} \leqslant f_X\left(\frac{\kappa_X}{\kappa_T}\right)\alpha_X^3. \tag{5.8}$$

By solving this for $1/\alpha_X$, by solving (3.23) for $1/\alpha$, and by substituting the results in (5.5) we get

$$1 \leqslant \left(\frac{\bar{S}'(0)}{\bar{T}'(0)}\right)^{\frac{2}{3}}\left(\frac{\kappa_S}{\kappa_T}\right)^{\frac{1}{3}} f^{\frac{1}{3}} + \left(\frac{\bar{X}'(0)}{\bar{T}'(0)}\right)^{\frac{2}{3}}\left(\frac{\kappa_X}{\kappa_T}\right)^{\frac{1}{3}} f_X^{\frac{1}{3}},$$

and the elimination of f_X by (5.7) yields

$$1 \leqslant \left(\frac{\bar{S}'(0)}{\bar{T}'(0)}\right)^{\frac{2}{3}}\left(\frac{\kappa_S}{\kappa_T}\right)^{\frac{1}{3}} f^{\frac{1}{3}} + \left(\frac{\bar{X}'(0)}{\bar{T}'(0)}\right)^{\frac{2}{3}}\left(\frac{\kappa_X}{\kappa_T}\right)^{\frac{1}{3}} (1-f)^{\frac{1}{3}} \equiv G(f). \tag{5.9}$$

Consider this function $G(f) = af^{\frac{1}{3}} + b(1-f)^{\frac{1}{3}}$, where (a, b) are the coefficients in (5.9). Since $G''(f) < 0$, G has a single extremum at $f = (1+(b/a)^{\frac{3}{2}})^{-1}$, the value of which is given by

$$G(f) \leqslant a[1+(b/a)^{\frac{3}{2}}]^{-\frac{1}{3}} + b(b/a)^{\frac{1}{2}}[1+(b/a)^{\frac{3}{2}}]^{-\frac{1}{3}}.$$

Further simplification gives $G(f) \leqslant (a^{\frac{3}{2}} + b^{\frac{3}{2}})^{\frac{2}{3}}$, and consequently (5.9) becomes

$$\frac{\bar{S}'(0)}{\bar{T}'(0)}\left(\frac{\kappa_S}{\kappa_T}\right)^{\frac{1}{2}} + \frac{\bar{X}'(0)}{\bar{T}'(0)}\left(\frac{\kappa_X}{\kappa_T}\right)^{\frac{1}{2}} \geqslant 1. \tag{5.10}$$

In correspondence with (2.3*a*, *b*) we also have

$$-\bar{X}'(z) = \frac{F_X - \overline{w_0 X_0}}{\kappa_X},$$

where $\overline{w_0 X_0}$ is the solute convective flux, and F_X is the constant total flux. By eliminating the gradients in (5.10) we get

$$\frac{F_S}{\kappa_S^{\frac{1}{2}}} + \frac{F_X}{\kappa_X^{\frac{1}{2}}} \geqslant \frac{F_T}{\kappa_T^{\frac{1}{2}}} + \left[\frac{\overline{w_0 S_0}}{\kappa_S^{\frac{1}{2}}} + \frac{\overline{w_0 X_0}}{\kappa_X^{\frac{1}{2}}} - \frac{\overline{w_0 T_0}}{\kappa_T^{\frac{1}{2}}}\right]_{z=0}. \tag{5.11}$$

The last step is the generalization of the entrainment inequality (2.4), which now implies that the total solute flux $\overline{w_0 S_0} + \overline{w_0 X_0}$ at (and below) $z = 0$ must exceed the heat flux $\overline{w_0 T_0}$. Equation (5.1) then implies that the bracketed term in (5.11) is positive, from which result we conclude that,

$$\frac{F_S}{\kappa_S^{\frac{1}{2}}} + \frac{F_X}{\kappa_X^{\frac{1}{2}}} \geqslant \frac{F_T}{\kappa_T^{\frac{1}{2}}}, \tag{5.12}$$

for all dynamically possible solutions. The realized one, according to our variational principle, is the bound

$$\frac{F_S}{\kappa_S^{\frac{1}{2}}} + \frac{F_X}{\kappa_X^{\frac{1}{2}}} = \frac{F_T}{\kappa_T^{\frac{1}{2}}}, \tag{5.12a}$$

provided the non-dimensional buoyancy flux is small (*cf.* (4.8*a*)).

6. Conclusions and comparisons

Our purpose has been to test some general ideas pertaining to turbulent boundary layers by examining the implications for the (rather complex) double diffusion problem.

The generalization of BLI implies that $(\kappa_S/\kappa_T)^{\frac{1}{2}}$ is the smallest possible salt/heat-flux ratio, and our variational principle implies that this bound should be realized when the total buoyancy flux is small (4.8*a*). BLI also predicts an upper bound (4.9) for the total buoyancy flux when $\Delta T \rightarrow \Delta S$. For triple diffusion the bound (5.12*a*) on the flux ratios should be realized when the buoyancy flux is small, and the prediction could be tested by future experiments which measure all three fluxes.

Double-diffusive measurements of r have been obtained using either heat/salt or salt/sugar, and it appears that the $(\kappa_S/\kappa_T)^{\frac{1}{2}}$ asymptote is realized near the 'cutoff' value of R, when the fluxes are small. The most accurate ($\pm 5\%$) determination of r is for the salt/sugar experiment, and it is remarkable that individual flux measurements 'scatter' very much more (this surely must be a real and physically significant statistical effect). Despite the latter fact some comparison with the present theory is possible.

For this purpose the bound (4.8*a*) is rewritten as

$$1-\frac{\tau}{r^2}=\left(\frac{r-\tau}{r-\tau R}\right)^4[0{\cdot}37F_T^*(1-r)]^3, \tag{6.1}$$

where

$$F_T^* \equiv \frac{gF_T\nu^{\frac{1}{3}}}{0{\cdot}085(g\Delta T)^{\frac{4}{3}}\kappa_T^{\frac{2}{3}}}$$

is a nominal non-dimensional flux which LS have used in their figure 4 to plot the data. Most of the latter have fluxes smaller than $F_T^* = 1{\cdot}0$, which value occurs at approximately $R = 1{\cdot}1$, $r = 0{\cdot}6$. The substitution of these three values and $\tau = \kappa_S/\kappa_T$ into the right-hand side of (6.1) gives a small number, 6×10^{-3}, from which the small departure

$$\frac{r-\tau^{\frac{1}{2}}}{\tau^{\frac{1}{2}}} \tag{6.2}$$

of r from its asymptotic value $\tau^{\frac{1}{2}}$ may be computed. Since the left-hand side of (6.1) equals twice (6.2), the value of the latter is 3×10^{-3}. This predicted departure is much smaller than the experimental error: 50×10^{-3}, but the two numbers become comparable when the largest measured flux $F_T^* \simeq 3$ is used. We conclude that the observed constancy of r in the sugar/salt experiments is consistent with the theory. In the heat/salt experiments of Turner (1973, figure 8.14) the largest 'reliable' flux is $F_T^* = 3$ at $R = 1{\cdot}3$, and from his figure (8.15) we see that the corresponding value of r is approximately 0·6 (although $r = 0{\cdot}15$ at larger R). These values and $\tau = \kappa_S/\kappa_T = 0{\cdot}01$ make the left-hand side of (6.1) equal to 0·98, which is indeed larger than the right-hand side (= 0·09). Although this is consistent with BLI (4.8), the bound predicted by the first-order variational theory is not reached by an order of magnitude.

It is possible to obtain the main results (3.26), (5.12*a*) by a simple mechanistic theory (LS) based on an adaption of Howard's argument for the equilibrium thickness of the thermal boundary layer in ordinary turbulent convection. This helpful theory was advanced provisionally because of internal inconsistencies. The mechanistic theory also says too much, since some of its equally plausible implications do not agree with experiments. Such conflicts are absent from the present theory, the reason being that it says so little (despite the ad hoc qualitative assertions used in constructing BLI).

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REFERENCES

DEARDORFF, J. W., WILLIS, G. E. & STOCKTON, B. H. 1980 Laboratory studies of the entrainment zone in a convectively mixed layer. *J. Fluid Mech.* **100**, 41–64.

GRIFFITHS, R. W. 1979 The transport of multiple components through thermohaline diffusive interfaces. *Deep-Sea Res.* **26**A, 383–397.

HUPPERT, H. E. & MOORE, D. R. 1976 Nonlinear double-diffusive convection. *J. Fluid Mech.* **78**, 821–854.

Linden, P. F. & Shirtcliffe, T. G. L. 1978 The diffusive interface in double-diffusive convection. *J. Fluid Mech.* **87**, 417–432.

Stern, M. E. 1975 *Ocean Circulation Physics*. Academic.

Stern, M. E. 1979 Inequalities and variational principles in turbulent transport. *J. Fluid Mech.* **91**, 513–540.

Stern, M. E. 1980 A variational principle for turbulent flow. *Phys. Fluids*, **23**, 2161–2170.

Turner, J. S. 1973 *Buoyancy Effect in Fluids*. Cambridge University Press.

Turner, J. S., Shirtcliffe, T. G. L. & Brewer, P. G. 1970 Elemental variations of transport coefficients across density surfaces in multiple diffusive systems. *Nature* **228**, 1083–1084.

Veronis, G. 1968 Effect of a stabilizing gradient of solute on thermal convection. *J. Fluid Mech.* **34**, 315–336.